

# Geometric phase in dephasing systems

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Beyond the quantum Markov approximation, we calculate the geometric phase of a two-level system driven by a quantized magnetic field subject to phase dephasing. The phase reduces to the standard geometric phase in the weak coupling limit and it involves the phase information of the environment in general. In contrast with the geometric phase in dissipative systems, the geometric phase acquired by the system can be observed on a long time scale. We also show that with the system decohering to its pointer states, the geometric phase factor tends to a sum over the phase factors pertaining to the pointer states.

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Quantum mechanics tell us that physical states are equivalent up to a global phase, which in general does not contain useful information about the described system and thus can be ignored. This is not the case, however, for a system transported round a circuit by varying the parameters  $\vec{s} = (s_1, s_2, \dots)$  in its Hamiltonian  $H(\vec{s})$ . As Berry showed [1], the phase can have a component of geometric origin called geometric phase with important observable consequences, such as the Aharonov-Bohm effect [2] and the spin- $\frac{1}{2}$  particle driven by a rotating magnetic field [1]. The geometric phases that only depend on the path followed by the system during its evolution, have been investigated and tested in a variety of settings and have been generalized in several directions [3]. The geometric phases are attractive both from a theoretical perspective, and from the point of view of possible applications, among which geometric quantum computation [4, 5, 6, 7] is one of the most importance.

As realistic systems always interact with their environment, the study on the geometric phase in open systems become interesting. Garrison and Wright [8] were the first to touch on this issue by describing open system evolution in terms of a non-Hermitian Hamiltonian. This is a pure state analysis, so it did not address the problem of geometric phases for mixed states. Toward the geometric phase for mixed states in open systems, the approaches used involve solving the master equation of the system [9, 10, 11, 12, 13], employing a quantum trajectory analysis [14, 15] or Krauss operators [16], and the perturbative expansions [17, 18]. Some interesting results were achieved, briefly summarized as follows: nonhermitian Hamiltonian lead to a modification of Berry's phase [8, 17], stochastically evolving magnetic fields produce both energy shift and broadening [18], phenomenological weakly dissipative Liouvillians alter Berry's phase by introducing an imaginary correction [11] or lead to damping and mixing of the density matrix elements [12]. However, almost all these studies are performed for dissipative systems, and thus the representations are applicable for systems whose energy is not conserved. For open systems with conserved energy (dephasing systems), the

problem beyond the Markov approximation remains untouched to our best knowledge. Because the system-environment interaction  $H_I$  and the free system Hamiltonian  $H_s$  commute for dephasing systems, the dynamical problem and then the geometric phase of the system may be solved/calculated precisely. On the other hand, the previous study [17] shows that one can not perform an arbitrarily long experiment to measure the phase for dissipative systems, i.e., it is not allowed to draw phase information out of the system on a long time scale, this feature of dissipative systems again motivates investigation on the problem of geometric phases in dephasing systems, where in principle one may get analytical results for the phase on any time scale.

In this Letter, we investigate the behavior of the geometric phase of a two-level system interacting with a driving magnetic field when this system is not only quantized but also subjected to decoherence. The environment that leads to decoherence may originate from the fluctuation in the driving fields, or from the vacuum fluctuations, or from the background radiations. We calculate and analyze the effect of dephasing of the driven system on the geometric phase of the system, our discussions will distinguish between two kinds of evolution: (1) The environment undergoes an adiabatic evolution, and (2) it evolves as it may.

Let us consider a two-level system driven by a quantized magnetic field and subjected to decoherence. The decoherence process is described by the coupling of the two-level system to an environment of harmonic oscillators with frequencies  $\{\omega_j\}$ , the Hamiltonian governing such a system reads [19]

$$\begin{aligned} H &= H_s + H_I + H_e, \\ H_s &= \hbar \frac{\Omega}{2} \sigma_z + \hbar \Omega a^\dagger a + \hbar g(\sigma_+ a + \sigma_- a^\dagger), \\ H_I &= (\sigma_+ a + \sigma_- a^\dagger) \sum_j \hbar \lambda_j (b_j^\dagger + b_j), \\ H_e &= \hbar \sum_j \omega_j b_j^\dagger b_j, \end{aligned} \quad (1)$$

where  $a, a^\dagger$  are boson operators for the driving field,  $\sigma_+$ ,  $\sigma_-$ , and  $\sigma_z$  are pauli operators for two relevant internal atomic levels  $|e\rangle$  and  $|g\rangle$ , and  $b_j^\dagger, b_j$  are the creation and annihilation operators of the environment bosons. The system hamiltonian  $H_s$  characterizes Jaynes-Cummings dynamics without neither energy relaxation nor phase dephasing, while terms  $H_e$  and  $H_I$  describe a bosonic environment and its coupling to the Jaynes-Cummings system. The choice of the coupling between the system and the environment determines effects of the environment. For example, the choice of the system operators that do not change the quantum number of the driving field when they operate on the dressed state would result in relaxations within the dressed states indicated by the quantum number  $n$ , but not energy relaxations between states with different  $n$ . The system-environment coupling chosen in our model is exactly such a choice, and thus, we may rewrite the Hamiltonian Eq. (1) in terms of the dressed states as

$$\begin{aligned} H &= \bigoplus_n H_n, \\ H_n &= E_+(n)|+(n)\rangle\langle+(n)| \\ &\quad + E_-(n)|-(n)\rangle\langle-(n)| + \sum_j \hbar\omega_j b_j^\dagger b_j \\ &\quad + \sqrt{n+1}(|+(n)\rangle\langle+(n)| \\ &\quad - |-(n)\rangle\langle-(n)|) \sum_j \hbar\lambda_j (b_j^\dagger + b_j), \end{aligned} \quad (2)$$

where  $E_\pm(n) = \frac{2n+1}{2}\hbar\Omega \pm \hbar g\sqrt{n+1}$ , and the dressed states with indication  $n$  are

$$|\pm(n)\rangle = \frac{1}{\sqrt{2}}(|g, n+1\rangle \pm |e, n\rangle). \quad (3)$$

Here  $\{|n\rangle\}$  stand for the Fock states of the driving field. The Hamiltonian  $H_n$  describes a driven harmonic oscillator, the driving terms depend on the dressed states, but they are independent of the dressed state indication  $n$ . With these properties, we may simplify the problem and restrict our study within the dressed states with the same indication  $n$ . The eigenstates of the Hamiltonian  $H_n$  take the following form,

$$\begin{aligned} |e_+(n)\rangle_{\{n_j\}} &= |+(n)\rangle \otimes \prod_j |n_j\rangle_{B_{+,j}}, \\ |e_-(n)\rangle_{\{n_j\}} &= |-(n)\rangle \otimes \prod_j |n_j\rangle_{B_{-,j}}, \end{aligned} \quad (4)$$

where  $|n_j\rangle_{B_{\pm,j}}$  denotes the Fock state for new environment mode  $B_{\pm,j} = (b_j \pm \lambda_j\sqrt{n+1}/\omega_j)|\pm(n)\rangle\langle\pm(n)|$ , i.e.,  $B_{\pm,j}^\dagger B_{\pm,j}|n_j\rangle_{B_{\pm,j}} = n_j|n_j\rangle_{B_{\pm,j}}$ , depending on the dressed states. The corresponding eigenenergies are  $\hbar g\sqrt{n+1} + \sum_j \hbar\omega_j n_j$  and  $-\hbar g\sqrt{n+1} + \sum_j \hbar\omega_j n_j$ , respectively. We distinguish between two situations to study the geometric phase of the system, the first is

to consider the universe (system+environment) to undergo an adiabatic evolution, in this case the states of the environment never evolve during the evolution. Because the relaxation of the system due to its coupling to the environment is independent of the dressed state indication  $n$ , the acquired geometric phase of the open system are the same as in the case without the environment. The second situation is more practical, in which we assume the environment initially is in its ground state  $\prod_j |0_j\rangle_{b_j} = \prod_j |\pm \frac{\lambda_j\sqrt{n+1}}{\omega_j}\rangle_{B_{\pm,j}}^c$  and evolve govern by the Hamiltonian Eq. (1). This initial state means that the environment is initialized in the vacuum of modes  $\{b_j\}$ , or in coherent state  $|\pm \frac{\lambda_j\sqrt{n+1}}{\omega_j}\rangle_{B_{\pm,j}}^c$  of modes  $\{B_{\pm,j}\}$ . The geometric phase of the universe in this situation may be calculated by removing the accumulation of these dynamical phase from the total phase, i.e.,

$$\phi_g(n) = \arg\langle\Psi(0)|\Psi(T)\rangle + i \int_0^T dt \langle\Psi(t)|\frac{\partial}{\partial t}|\Psi(t)\rangle, \quad (5)$$

it is easy to demonstrated that for closed systems  $\phi_g(n)$  reduces to the Aharonov-Anandan formula for cyclic evolutions [20] and to the Berry phase for adiabatic and cyclic evolutions [1]. The initial state together with the time evolution operator determine the path followed by the system, in this sense the geometric phase might depend on the initial condition. If we choose  $|\Psi(0)\rangle = |+(n)\rangle \otimes \prod_j |\frac{\Lambda_j}{\omega_j}\rangle_{B_{+,j}}^c$  ( $\Lambda_j = \lambda_j\sqrt{n+1}$ ) as the initial state, at time  $t$  the universe evolves to

$$|\Psi(t)\rangle = |+(n)\rangle \otimes \prod_j |\frac{\Lambda_j}{\omega_j} e^{-i\omega_j t}\rangle_{B_{+,j}}^c. \quad (6)$$

In contrast with classically driving field, in this study the driving field is quantized. In order to generate a phase change in the state of the field, we borrow the idea in Ref.[21] to introduce the phase shift operator  $U(\psi) = \exp(-i\psi a^\dagger a)$  and adiabatically apply it to the Hamiltonian of the system. Changing  $\psi = \bar{\Omega} \cdot t$  slowly from 0 to  $2\pi$  (the corresponding time from 0 to  $T = \frac{2\pi}{\bar{\Omega}}$ ) the geometric phase generated is calculated by Eq. (5) as follows,

$$\begin{aligned} \phi_g^+(n) &= \arg[\prod_j e^{-|\frac{\Lambda_j}{\omega_j}|^2(1-e^{-i\omega_j T})}] \\ &\quad + \sum_j [\omega_j T e^{-|\frac{\Lambda_j}{\omega_j}|^2} \sum_{m=0}^{\infty} m \frac{(\Lambda_j/\omega_j)^{2m}}{m!}] \\ &\quad + \gamma_+(n), \end{aligned} \quad (7)$$

where  $\gamma_+(n) = (2n+1)\pi$  is the Berry phase acquired when the universe remains unchanged. For a continuous spectrum of the environmental modes, the sum over  $j$  in the above expressions is replaced by an integral involving the spectral density with a cutoff frequency  $\omega_c$

$$\rho(\omega) = \varepsilon(\frac{\omega}{\lambda_\omega})^2, 0 \leq \omega \leq \omega_c, \quad (8)$$

this spectrum density is of Ohmic type. Eq. (5) and Eq.(8) together yield

$$\phi_g^+(n) = \gamma_+(n) + \varepsilon \frac{\omega_c^2(n+1)T}{2} + \frac{(n+1)}{T} \varepsilon [\cos(\omega_c T) - 1]. \quad (9)$$

If we choose the eigenstate  $|\Psi(0)\rangle = |-(n)\rangle \otimes \prod_j |-\frac{\Lambda_j}{\omega_j}\rangle_{B-,j}^c$  from another set of eigenstates Eq. (4) as the initial condition, the time evolution of the universe can be expressed as,

$$|\Phi(t)\rangle = |-(n)\rangle \otimes \prod_j |-\frac{\Lambda_j}{\omega_j} e^{-i\omega_j t}\rangle_{B-,j}^c. \quad (10)$$

In the same way, we can get the geometric phase pertaining to this evolution loop,

$$\phi_g^-(n) = \phi_g^+(n) - \gamma_+(n) + \gamma_-(n), \quad (11)$$

namely, the contributions from the system-environment coupling are the same for the both pathes. Here  $\gamma_-(n)$  is the Berry phase attaining to the dressed state  $|-(n)\rangle$  in the case without environment, it is easy to prove that it takes the same expression as  $\gamma_+(n)$ .

The last two terms in equation Eq. (9) result from the system-environment couplings, they vanish when the couplings tend to zero (in the equations,  $\omega_c \rightarrow 0$ ), thus the expressions return to the geometric phases presented in Ref. [21]. The path the system followed changes the system-environment coupling, and the environment is impossible to return to its initial state due to its huge variety of freedom, the geometric phase then depends on the time  $T$  when we draw out phase information from the system. In the classical limit  $n \rightarrow \infty$ , the contributions from the system-environment coupling tend to infinity caused by relative strong coupling between the system and the driving field, hence it becomes undefined in the sense of interferometry. It is interesting to note that for  $n = 0$

the phases are not zero, which means that the vacuum driving field introduces a correction in geometric phases, this expression is relevant when systems are driven by fields with few photons. For weak system-environment coupling [22], Eq. (7)( Eq. (11), in the same way) may be expanded in powers of  $\Lambda_j/\omega_j$ , up to the second order of  $\Lambda_j/\omega_j$ , Eq.(7) follows,

$$\phi_g(n) = \gamma_+(n) + \sum_j \omega_j T \left(\frac{\Lambda_j}{\omega_j}\right)^2. \quad (12)$$

The explanation of this result is very simple, as the system-environment coupling is very weak, the most contribution to the geometric phase come from the system-driving field coupling. The same dependence of  $\phi_g(n)$  on  $\Lambda_j$  (i.e.,  $\Lambda_j^2$ , no contribution proportional to  $\Lambda_j$ ) can be found in Ref. [17].

Up to now, we have calculated the geometric phase for the universe, explicit expressions for the geometric phase were obtained, these expressions are of relevance for the case in which systems are coupled to an environment that describes parameter fluctuations. For instance, imperfect dipole transitions between states  $|g\rangle$  and  $|e\rangle$  due to fluctuation of the driving laser intensity may be modelled by the coupling in Eq. (1), in this situation the environment and the driving field are the same. This is not the case, however, when the environment is an independent system (say, black body radiations), we have to trace out the environment in order to calculate the dynamical information for the system, thus the evolution of the system is no longer unitary. For non-unitary evolution, the geometric phase can be calculated as follows. First, solve the eigenvalue problem for the reduced density matrix  $\rho(t)$  and obtain its eigenvalues  $\varepsilon_k(t)$  as well as the corresponding eigenvectors  $|\psi_k(t)\rangle$ ; secondly, substitute  $\varepsilon_k(t)$  and  $|\psi_k(t)\rangle$  into

$$\Phi_g(n) = \arg\left(\sum_k \sqrt{\varepsilon_k(0)\varepsilon_k(T)} \langle\psi_k(0)|\psi_k(T)\rangle e^{-\int_0^T \langle\psi_k(t)|\partial/\partial t|\psi_k(t)\rangle dt}\right). \quad (13)$$

Here,  $\Phi_g(n)$  is the geometric phase for the system undergoing non-unitary evolution [23]. The geometric phase Eq. (13) is gauge invariant and can be reduced to the well-known results in the unitary evolution, thus it is experimentally testable. Now, we exploit the expression to calculate the geometric phase for the driven two-level system. To this aim, we first write down the reduced

density matrix  $\rho(t)$ ,

$$\rho(t) = \begin{pmatrix} |c_+|^2 & c_+^* c_- F(t) \\ c_+ c_-^* F^*(t) & |c_-|^2 \end{pmatrix}, \quad (14)$$

where the initial state of  $(c_+| + (n)\rangle + c_-| - (n)\rangle) \otimes \prod_j |0_j\rangle_{b_j}$  is assumed for the universe, and  $F(t) = \exp[-\frac{i}{\hbar}(E_+(n) - E_-(n))t] \cdot \exp[-\sum_j \eta_j(t)]$  with  $\eta_j(t) = 4|\frac{\Lambda_j}{\hbar\omega_j}|^2(1 - \cos \omega_j t)$ . Simple algebra gives the eigenvalues

and the corresponding eigenvectors of  $\rho(t)$ ,

$$\begin{aligned}\varepsilon_{\pm}(t) &= \frac{1}{2} \pm \frac{1}{2} \sqrt{(|c_+|^2 - |c_-|^2)^2 + 4|c_+^* c_- F(t)|^2}, \\ |\psi_{\pm}(t)\rangle &= X_{\pm}(t)|+(n)\rangle + Y_{\pm}(t)|-(n)\rangle,\end{aligned}\quad (15)$$

where  $X_{\pm}(t) = c_+^* c_- F(t) / \sqrt{|c_+^* c_- F(t)|^2 + (\varepsilon_{\pm}(t) - |c_+|^2)}$ , and  $Y_{\pm}(t) = \sqrt{1 - |X_{\pm}(t)|^2}$ . Eq. (15) and Eq.(13) together yield the geometric phase for the system. The expression for the geometric phase is tedious, so instead of writing down the expression, we present here some remarkable comments. The expressions for the geometric phase are analytically exact, so we might predict the behavior of the geometric phase on a long time scale. For any  $j$ ,  $\eta_k(t) \geq 0$ , so with  $t \rightarrow \infty$ ,  $F(t) \rightarrow 0$  for a random spectrum density of the environment, this indicates that  $\Phi_g(T \rightarrow \infty) \rightarrow \arg[|c_+|^2 e^{-i\gamma_+(n)} + |c_-|^2 e^{-i\gamma_-(n)}]$  for a relative long  $T$ . This result is quite interesting: With the off-diagonal elements of the reduced density matrix tending to zero (decoherence), the driven two-level system decoheres to its pointer states  $|+(n)\rangle$  or  $|-(n)\rangle$ , while the geometric phase factor of the driven system reduces to a weighted sum over the phase factors pertaining to the pointer states, this provides us a new way to observe decoherence effects.

Summarizing, the geometric phases for a dephasing system (open system) have presented and discussed. The open system has been demonstrated by a driven two-level system coupling to an environment of harmonic oscillators. The results show that there are no correction in the geometric phase of the universe due to the system-environment coupling when the environment undergoes an adiabatic evolution, whereas the correction in the phase is path dependent when the constraint on the evolution is released. The mixed state geometric phase for the dephasing system is also presented and discussed. The geometric phase factor would tend to a sum over these phase factor pertaining to the pointer states with the open system decohering to its pointer states, it is a reflection of the decoherence in geometric phases.

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- [1] M. V. Berry, Proc. R. Soc. London A **392**, 45(1984).
  - [2] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485(1959).
  - [3] Geometric phase in physics, Edited by A. Shapere and F. Wilczek ( World Scientific, Singapore, 1989).
  - [4] P. Zanardi and M. Rasetti, Phys. Lett. A **264**, 94 (1999).
  - [5] J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature (London) **403**, 869 (1999).
  - [6] A. Ekert, M. Ericsson, P. Hayden, H. Inamori, J.A. Jones, D.K.L. Oi, and V. Vedral, J. Mod. Opt. **47**, 2051 (2000).
  - [7] G.Falci, R. Fazio, G.M. Palma, J. Siewert, and V. Vedral, Nature (London) **407**, 355 (2000).
  - [8] J. C. Garrison and E. M. Wright, Phys. Lett. A **128**, 177(1988).
  - [9] K. M. Fonseca Romero, A. C. Aguirra Pinto, and M. T. Thomaz, Physica A **307**, 142(2002).
  - [10] A. C. Aguirra Pinto and M. T. Thomaz, J. Phys. A: Math. Gen. **36**, 7461(2003).
  - [11] D. Ellinas, S. M. Barnett, and M. A. Dupertuis, Phys. Rev. A **39**, 3228(1989).
  - [12] D. Gamliel and J. H. Freed, Phys. Rev. A **39**, 3238(1989).
  - [13] I. Kamleitner, J. D. Cresser, and B. C. Sanders, Phys. Rev. A **70**, 044103(2004).
  - [14] A. Nazir, T. P. Spiller, W. J. Munro, Phys. Rev. A **65**, 042303(2002).
  - [15] A. Carollo, I. Fuentes-Guridi, M. Franca Santos and V. Vedral, Phys. Rev. Lett. **90**,160402(2003); *ibid* **92**, 020402(2004).
  - [16] K. P. Marzlin, S. Ghose, and B. C. Sanders, e-print:quant-ph/0405052(Phys. Rev. Lett., to be published).
  - [17] R. S. Whitney, and Y. Gefen, Phys. Rev. Lett. **90**, 190402(2003); R. S. Whitney, Y. Makhlin, A. Shnirman, and Y. Gefen, e-print:cond-mat/0405267.
  - [18] F. Gaitan, Phys. Rev. A **58**, 1665(1998).
  - [19] S. Bose, P. L. Knight, M. Muraio, M. B. Plenio, and V. Vedral, Phil. Trans. Roy. Soc. Lond. A **356**, 1823(1998).
  - [20] Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593(1987).
  - [21] I. Fuentes-Guridi, A. Carollo, S. Bose, and V. Vedral, Phys. Rev. Lett. **89**, 220404(2002).
  - [22] The weak coupling means that  $\Lambda_j/\omega_j \ll 1$ , the system-environment couplings fall in this regime when there are few quanta in the driving field, and  $\lambda_j$  is adjusted small with respect to  $\omega_j$ . This is possible for the engineered environment in trapped ion system, see for example, C. J. Myatt, *et al.*, Nature **403**, 1269(2000).
  - [23] D. M. Tong, E. Sjöqvist, L. C. Kwek, C. H. Oh, Phys. Rev. Lett. **93**, 080405 (2004).